

# STATISTICAL PROPERTIES OF NONUNIFORMLY EXPANDING 1D MAPS WITH LOGARITHMIC SINGULARITIES

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**ABSTRACT.** For a certain parametrized family of maps on the circle, with critical points and logarithmic singularities where derivatives blow up to infinity, a positive measure set of parameters was constructed in [19], corresponding to maps which exhibit nonuniformly hyperbolic behavior. For these parameters, we prove the existence of absolutely continuous invariant measures with good statistical properties, such as exponential decay of correlations. Combining our construction with the logarithmic nature of the singularities, we obtain a positive variance in Central Limit Theorem, for any nonconstant Hölder continuous observable.

## 1. INTRODUCTION

Let  $f_{a,L} : \mathbb{R} \rightarrow \mathbb{R}$  be such that

$$(1) \quad f_{a,L} : x \mapsto x + a + L \ln |\Phi(x)|,$$

where  $a \in [0, 1]$ ,  $L \in \mathbb{R}$  are real parameters and  $\Phi(x)$  is such that  $\Phi(x+1) = \Phi(x)$ . We assume that  $\Phi(x)$  is a Morse function, the graph of  $y = \Phi(x)$  intersects  $x$ -axis, and all the intersections are transverse. The functions  $f_{a,L}$  induce a two parameter family of endomorphisms on  $S^1 = \mathbb{R}/\mathbb{Z}$ , having non-degenerate critical points and singularities where the value of  $f_{a,L}$  is undefined. For sufficiently large  $|L|$ , a positive measure set  $\Delta(L)$  of the parameter  $a$  was constructed in [19], such that if  $a \in \Delta(L)$ , then  $f_{a,L}$  admits an invariant measure that is absolutely continuous with respect to Lebesgue measure (acim). In this paper we study statistical properties of this measure.

This class of systems is motivated by the recent studies [23, 24, 25] on homoclinic tangles and strange attractors in periodically forced differential equations ( $S^1$  reflects the time-periodicity of the force). In brief terms, the maps  $f_{a,L}$  as we treat here can be obtained by considering first-return maps of the flow (in the extended phase space introducing the time as a new variable) to appropriate cross-sections, and then passing to a singular limit. This last step results in a considerable simplification of the dynamics. Nevertheless, the map  $f_{a,L}$  retains a large share of the complexity of the corresponding flow, and thus, provide an important insight to its behavior.

Apart from this original motivation, the family of circle maps is of interest in its own light, for the feature of the logarithmic singularities that turns out to influence on some statistical properties of the acims, as we explain in the sequel.

**1.1. Statements of the results.** For smooth maps on the interval or the circle, it is now classical that an exponential growth of derivatives along the orbits of critical points implies the existence of acims with good statistical properties [1, 4, 10, 26, 27]. Our first result is a

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version of this for  $f_{a,L}$  with critical and singular points. Dynamical properties shared by maps corresponding to parameters in  $\Delta(L)$  are listed in Section 2.4.

**Theorem A.** *For any  $f \in \{f_{a,L} : a \in \Delta(L)\}$  there exists an ergodic  $f$ -invariant probability measure  $\mu$  that is equivalent to Lebesgue measure. In addition,*

(1) *for any  $\eta \in (0, 1]$  there exists  $\tau \in (0, 1)$  such that for any Hölder continuous function  $\varphi$  on  $S^1$  with Hölder exponent  $\eta$  and  $\psi \in L^\infty(\mu)$ , there exists a constant  $K(\varphi, \psi)$  such that*

$$\left| \int (\varphi \circ f^n) \psi d\mu - \int \varphi d\mu \int \psi d\mu \right| \leq K(\varphi, \psi) \tau^n \quad \text{for every } n > 0;$$

(2)  *$(f, \mu)$  satisfies Central Limit Theorem (see the definition below).*

This is not the first result on statistical properties of one-dimensional maps with critical and singular points. A certain class of maps were studied in [6], including the Lorenz-like maps corresponding to positive measure sets of parameters constructed in [13, 14]. Maps with singularities and infinitely many critical points were studied in [17]. To our knowledge, however, there is no previous study on statistical properties of maps with logarithmic singularities. For instance, one key aspect of our maps that has no analogue in those of the previous studies is that, returns to a neighborhood of singularities can happen very frequently. The previous arguments seem not sufficient to deal with points like this.

In the study of dynamical systems with singularities, influences of singularities on dynamics are not well understood. Indeed, singularities with blowing up derivatives help to create expansion, and to enforce a chaotic behavior. However, little is known on influences of singularities on statistical properties of the systems. In this direction, one result we are aware is [12] which takes advantage of the singularity of the expanding Lorenz map to show that the Lorenz attractor is mixing. In the proof of Theorem A, we design our construction in such a way that allows us to draw a new conclusion on Central Limit Theorem, viewed as an influence of the logarithmic singularities.

Let  $g: X \rightarrow X$  be a dynamical system preserving a probability measure  $\nu$ . We say  $(g, \nu)$  satisfies Central Limit Theorem if for any Hölder continuous function  $\phi$  on  $X$  with  $\int \phi d\nu = 0$ ,

$$\frac{1}{\sqrt{n}} \sum_{i=0}^{n-1} \phi \circ g^i \longrightarrow \mathcal{N}(0, \sigma) \quad \text{in distribution,}$$

where  $\mathcal{N}(0, \sigma)$  is the normal distribution with mean 0 and variance  $\sigma^2$ , and

$$\sigma^2 = \lim_{n \rightarrow \infty} \frac{1}{n} \int \left( \sum_{i=0}^{n-1} \phi \circ g^i \right)^2 d\nu.$$

If  $\sigma > 0$ , this means that for every interval  $J \subset \mathbb{R}$ ,

$$\nu \left\{ x \in X : \frac{1}{\sqrt{n}} \sum_{i=0}^{n-1} \phi(g^i(x)) \in J \right\} \longrightarrow \frac{1}{\sigma\sqrt{2\pi}} \int_J e^{-\frac{t^2}{2\sigma^2}} dt.$$

It is known (see e.g. [18, 27]) that  $\sigma > 0$  if and only if  $\phi$  is *not coboundary*, that is, the cohomological equation

$$\phi = \psi \circ g - \psi$$

has no solution in  $L^2(\nu)$ . Otherwise,  $\phi$  is called *coboundary*. For dynamical systems satisfying Central Limit Theorem, determining the largest possible classes of functions which are not coboundary is an intricate problem, even for Axiom A systems [18]. For countable Markov maps on intervals, Morita [16] obtained Central Limit Theorem for a broad class of functions including those with bounded variation, and proved that there exists no non-trivial function which is coboundary. Our construction in Theorem A and the nature of the singularities allow us to show that, for our maps, there exists no non-trivial Hölder continuous function which is coboundary.

**Theorem B.** *Let  $(f, \mu)$  be as above. If a Hölder continuous  $\phi: S^1 \rightarrow \mathbb{R}$  with  $\int \phi d\mu = 0$  is coboundary, then  $\phi \equiv 0$ .*

Our strategy in Theorem A is to construct an induced Markov map and apply the scheme of Young [27]. A key feature of this construction is that the domain of the induced Markov map is a full measure subset of  $S^1$ . In other words,  $S^1$  is cut into pieces, and each piece grows to the entire  $S^1$  in a controlled way. As a consequence,  $\mu$  is equivalent to Lebesgue.

A proof of Theorem B is outlined as follows. Suppose that  $\phi$  is coboundary, with an  $L^2$  solution  $\psi$ . Using the induced Markov map, it is possible to show that  $\psi$  has a version  $\tilde{\psi}$  (i.e.  $\psi = \tilde{\psi}$   $\mu$ -a.e.) which is (Hölder) continuous on the entire  $S^1$ . On the other hand, the distinctive property of the logarithmic singularities is that, a small neighborhood of a singular point is divided into a countable number of intervals, and each of them is sent to the entire  $S^1$  just by one iterate. This property allows us to rule out the existence of nonconstant continuous solution of the cohomological equation. Hence,  $\tilde{\psi}$  has to be a constant function, and  $\phi \equiv 0$  follows.

The rest of this paper consists of four sections. In Section 2 we collect necessary materials in [19] as far as we need them. In Section 3 we perform a large deviation argument, a key step for the construction of the induced Markov map. In Section 4 we put these results together and construct an induced Markov map with exponential tails, and prove the theorems. In Section 5 we prove an entropy formula, connecting the metric entropy to the Lyapunov exponent.

## 2. PROPERTIES OF NONUNIFORMLY EXPANDING MAPS

This section collects materials in [19] as far as we need them. Dynamical properties shared by maps corresponding to the parameters in  $\Delta(L)$  are stated in Section 2.4.

**2.1. Elementary facts.** From this point on we use  $L$  for both  $L$  and  $|L|$ . We take  $L$  as a base of the logarithmal function  $\log(\cdot)$ . For  $f = f_{a,L}$ , let  $C(f) = \{f'(x) = 0\}$  denote the set of critical points and  $S(f) = \{\Phi(x) = 0\}$  the set of singular points. The distances from  $x \in S^1$  to  $C(f)$  and  $S(f)$  are denoted by  $d_C(x)$  and  $d_S(x)$  respectively. For  $\varepsilon > 0$ , we use  $C_\varepsilon$  and  $S_\varepsilon$  to denote the  $\varepsilon$ -neighborhoods of  $C$  and  $S$  respectively.

**Lemma 2.1.** [[19] Lemma 1.1.] *There exists  $K_0 > 1$  and  $\varepsilon_0 > 0$ , such that for all  $L$  sufficiently large and  $f = f_{a,L}$ ,*

(a) *for all  $x \in S^1$ ,*

$$K_0^{-1}L \frac{d_C(x)}{d_S(x)} \leq |f'x| \leq K_0L \frac{d_C(x)}{d_S(x)}, \quad |f''x| \leq \frac{K_0L}{d_S^2(x)};$$

(b) *for all  $\varepsilon > 0$  and  $x \notin C_\varepsilon$ ,  $|f'x| \geq K_0^{-1}L\varepsilon$ ; and*

(c) *for all  $x \in C_{\varepsilon_0}$ ,  $K_0^{-1}L < |f''x| < K_0L$ .*

*Sketch of the proof.* Use the assumptions on  $\Phi(x)$ :  $\Phi'(x) \neq 0$  on  $\{\Phi(x) = 0\}$ ;  $\Phi''(x) \neq 0$  on  $\{\Phi'(x) = 0\}$ .  $\square$

**2.2. Bounded distortion.** For  $x \in S^1$ ,  $n \geq 1$ , let

$$(2) \quad D_n(x) = \frac{1}{\sqrt{L}} \cdot \left[ \sum_{0 \leq i < n} d_i^{-1}(x) \right]^{-1} \quad \text{where} \quad d_i(x) = \frac{d_C(f^i x) \cdot d_S(f^i x)}{|(f^i)'x|},$$

when they make sense.

**Lemma 2.2.** *The following holds for all sufficiently large  $L$ : if  $n \geq 1$  and  $x, fx, \dots, f^{n-1}x \notin C \cup S$ , then for all  $\xi, \eta \in [x - D_n(x), x + D_n(x)]$ ,*

$$\frac{|(f^n)'\xi|}{|(f^n)'\eta|} \leq 2 \quad \text{and} \quad \left| \frac{|(f^n)'\xi|}{|(f^n)'\eta|} - 1 \right| \leq \frac{1}{L^{1/3}} \frac{|f^n \xi - f^n \eta|}{D_n(x) |(f^n)'x|}.$$

**Remark 2.1.** In Section 3 we will use these estimates on a bigger interval (comparable in length), but this does not seriously affect the estimates.

*Proof.* The first estimate was in [[19] Lemma 1.1]. We prove the second one. Let  $I$  denote the subinterval of  $[x - D_n(x), x + D_n(x)]$  with endpoints  $\xi, \eta$ . Let  $i \in [0, n)$ . By Lemma 2.1, for any  $\phi \in f^i I$ ,

$$\frac{|f''\phi|}{|f'\phi|} \leq \frac{K_0^2}{d_C(\phi)d_S(\phi)} \leq \frac{2K_0^2}{d_C(f^i x)d_S(f^i x)},$$

where the last inequality follows from

$$|f^i I| \leq 2D_n(x)|(f^i)'x| \leq \frac{2}{\sqrt{L}} d_C(f^i x) \cdot d_S(f^i x) \ll \max\{d_C(f^i x), d_S(f^i x)\}.$$

We also have

$$|f^i I| \leq 2|(f^i)'x||\xi - \eta|d_i(x)d_i^{-1}(x) = 2|\xi - \eta|d_C(f^i x) \cdot d_S(f^i x)d_i^{-1}(x).$$

Multiplying these two inequalities,

$$|f^i I| \sup_{\phi \in f^i I} \frac{|f''\phi|}{|f'\phi|} \leq 4K_0^2 |\xi - \eta| d_i^{-1}(x).$$

Summing this over all  $0 \leq i < n$  we obtain

$$\begin{aligned} \ln \frac{|(f^n)'\xi|}{|(f^n)'\eta|} &\leq \sum_{0 \leq i < n} \ln \frac{|f'(f^i \xi)|}{|f'(f^i \eta)|} \leq \sum_{0 \leq i < n} |f^i I| \sup_{\phi \in f^i I} \frac{|f''\phi|}{|f'\phi|} \\ &\leq \frac{4K_0^2 |\xi - \eta|}{\sqrt{L} D_n(x)} \leq \frac{8K_0^2 |f^n \xi - f^n \eta|}{\sqrt{L} D_n(x) |(f^n)'x|} \leq \frac{1}{L^{1/3}} \frac{|f^n \xi - f^n \eta|}{D_n(x) |(f^n)'x|}. \end{aligned}$$

The desired inequality holds.  $\square$

**2.3. Uniform expansion outside of critical regions.** Let  $\lambda = 10^{-3}$ ,  $\alpha = 10^{-6}$  and  $\delta = L^{-\alpha N_0}$ , where  $N_0$  is a large integer. Let  $\sigma = L^{-\frac{1}{6}}$ . For  $c \in C(f)$ , let  $v_0 = f(c)$  and  $\{v_i = f^i v_0, i \in \mathbb{Z}^+\}$ .

**Lemma 2.3.** [[19] Lemma 1.3.] *There exists a large integer  $N_0$  such that the following holds for all sufficiently large  $L$ : assume for each  $c \in C(f)$  and every  $0 \leq n \leq N_0$ ,  $d_C(v_n) \geq \sigma$  and  $d_S(v_n) \geq \sigma$ , then:*

- (a) *if  $n \geq 1$  and  $x, fx, \dots, f^{n-1}x \notin C_\delta$ , then  $|(f^n)'x| \geq \delta L^{2\lambda n}$ ;*
- (b) *if moreover  $f^n x \in C_\delta$ , then  $|(f^n)'x| \geq L^{2\lambda n}$ .*

*Sketch of the proof.* Let  $\delta_0 = L^{-\frac{11}{12}} \gg \delta$ . By Lemma 2.1, derivatives grow exponentially, as long as orbits stay outside of  $C_{\delta_0}$ . Once they fall in  $C_{\delta_0} \setminus C_\delta$ , they copy the growth of the derivatives of the nearest critical orbit for a certain period of time. The choice of  $\delta$  and the assumption on  $C(f)$  together ensure that this period is enough to recover an exponential growth.  $\square$

**2.4. Dynamical assumptions.** For the rest of this paper, we assume that  $N_0, L$  are large so that the conclusions of the previous three lemmas hold. In addition, for each  $c \in C(f)$  we assume:

- (a) for  $0 \leq n \leq N_0$ ,  $d_C(v_n) > \sigma$ ,  $d_S(v_n) > \sigma$ ;
- (b) for every  $n > N_0$ ,
  - (G1)  $|(f^{j-i})'v_i| \geq L \min\{\sigma, L^{-\alpha i}\}$  for every  $0 \leq i < j \leq n+1$ ;
  - (G2)  $|(f^i)'v_0| \geq L^{\lambda i}$  for every  $0 < i \leq n+1$ ;
  - (G3)  $d_S(v_i) \geq L^{-4\alpha i}$  for every  $N_0 \leq i \leq n$ .

Building on these standing assumptions, we construct an induced Markov map and deduce the properties in the theorems. It was proved in [19] that there exist a large integer  $N_0$  and  $L_0 > 0$  such that for all  $L \geq L_0$ , there exists a set  $\Delta(L) \subset [0, 1)$  of the parameter  $a$  with positive Lebesgue measure, such that (a) (b) hold for all  $f \in \{f_{a,L} : a \in \Delta(L)\}$ . In addition,  $\lim_{L \rightarrow \infty} |\Delta(L)| \rightarrow 1$  holds.

**2.5. Recovering expansion.** Let us introduce bound periods and recovery estimates from small derivatives near the critical set. Let  $c \in C$  and  $v_0 = f(c)$ . For  $p \geq 2$ , let

$$I_p(c) = \left( c + \sqrt{D_p(v_0)/(K_0 L)}, c + \sqrt{D_{p-1}(v_0)/(K_0 L)} \right].$$

Let  $I_{-p}(c)$  be the mirror image of  $I_p(c)$  with respect to  $c$ .

If  $x \in I_p(c) \cup I_{-p}(c)$ , then  $|fx - v_0| \leq D_{p-1}(v_0)$  holds. According to Lemma 2.2, the derivatives along the orbit of  $fx$  shadow that of the orbit of  $v_0$  for  $p-1$  iterates. We regard the orbit of  $x$  as bound to the orbit of  $c$  up to time  $p$ , and call  $p$  the *bound period* of  $x$  to  $c$ .

**Lemma 2.4.** [[19] Lemma 1.6.] *For every  $p \geq 2$  and  $x \in I_p(c) \cup I_{-p}(c)$ ,*

- (a)  $p \leq \log |c - x|^{-\frac{2}{\lambda}}$ ;
- (b) *if  $x \in C_\delta$ , then  $|(f^p)'x| \geq \max \left\{ |c - x|^{-1 + \frac{16\alpha}{\lambda}}, L^{\frac{\lambda}{3}p} \right\}$ .*

*Sketch of the proof.* (a) follows from the definition of  $D_p(v_0)$  and the assumption (G2) on  $v_0$ . The bounded distortion of  $f^{p-1}$  on  $f(I_p(c) \cup I_{-p}(c))$  and (G1), (G3) are used to prove (b).  $\square$

**2.6. Decomposition into bound/free segments.** We introduce a useful language along the way. Let  $x \in S^1 \setminus (C \cup S)$ . Let

$$0 \leq n_1 < n_1 + p_1 \leq n_2 < n_2 + p_2 \leq \cdots$$

be defined as follows:  $n_1$  is the smallest  $j \geq 0$  such that  $f^j x \in C_\delta$ , and called *the first return time of  $x$*  (even if it is 0). Given  $n_k$  with  $f^{n_k} x \in C_\delta$ ,  $p_k$  is the bound period and  $n_{k+1}$  is the smallest  $j \geq n_k + p_k$  such that  $f^j x \in C_\delta$ . This decompose the orbit of  $x$  into bound segments corresponding to time intervals  $(n_k, n_k + p_k)$  and free segments corresponding to time intervals  $[n_k + p_k, n_{k+1}]$ . The times  $n_k$  are called *free return times*.

**2.7. A few estimates.** We quote from [19] some technical estimates which will be used in Section 3. Let  $x \in S^1 \setminus (C \cup S)$  make a free return at  $\nu > 0$ . Let  $0 \leq n_1 < n_2 < \cdots < n_t < \nu$  denote all the free returns before  $\nu$ . Let  $p_1, p_2, \dots, p_t$  denote the corresponding bound periods. For each  $k \in [1, t]$ , let

$$\Theta_k(x) = \sum_{i=n_k}^{n_k+p_k-1} d_i^{-1}(x) \quad \text{and} \quad \Theta_0(x) = \sum_{i=0}^{\nu-1} d_i^{-1}(x) - \sum_{k=1}^t \Theta_k(x).$$

The quantity  $\Theta_k$  is the contribution of the bound segment from  $n_k$  to  $n_k + p_k - 1$  to the total distortion and  $\Theta_0$  is the contribution of all free segments to the total distortion. It is understood that if  $\nu$  is the first return time to  $C_\delta$ , then the second summand in the definition of  $\Theta_0(x)$  is 0.

The following two estimates were obtained in the proof of [[19] Lemma 1.8.], when  $x$  is a critical value. It is not hard to see that, the same estimates hold for a general  $x$ :

$$(3) \quad |(f^{n_k+p_k})'x|^{-1} \Theta_k(x) \leq |d_C(f^{n_k}x)|^{-\frac{18\alpha}{\lambda}};$$

$$(4) \quad |(f^\nu)'x|^{-1} \Theta_0(x) < \frac{1}{\delta^{\frac{1}{3}}}.$$

**Definition 2.1.** Let  $x \in S^1 \setminus (C \cup S)$ . We say  $\nu$  is a *deep return time* of  $x$  if it is the first return time of  $x$  to  $C_\delta$ , or else, for every free return  $n_k < \nu$ ,  $1 \leq k \leq t$ ,

$$(5) \quad 2 \log d_C(f^\nu x) + \sum_{n_j \in (n_k, n_t]} 2 \log d_C(f^{n_j} x) \leq \log d_C(f^{n_k} x).$$

We say  $\nu$  is a *shallow return time* of  $x$  if it is not a deep return time.

**Lemma 2.5.** Let  $\nu > 0$  be a deep return time of  $x \in S^1 \setminus (C \cup S)$ . Then

$$|(f^\nu)'x| \cdot D_\nu(x) \geq \sqrt{d_C(f^\nu x)}.$$

Lemma 2.2 gives a bounded distortion of  $f^\nu$  on the interval  $[x - D_\nu(x), x + D_\nu(x)]$ . Hence, Lemma 2.5 gives a lower estimate of the length of the interval  $f^\nu([x - D_\nu(x), x + D_\nu(x)])$  in terms of the distance of  $f^\nu(x)$  to the critical set. It follows that this interval contains a critical point to which  $f^\nu(x)$  is bound.

The following estimate, obtained in the proof of [[19] Proposition 2.1], bounds a contribution from shallow returns by that of deep returns.

**Lemma 2.6.** *Let  $0 \leq n_1 < n_2 < \dots < n_t$  denote all the free return times of  $x \in S^1 \setminus (C \cup S)$  up to time  $n_t$ . Then*

$$\sum_{n_1 \leq n_j \leq n_t: \text{shallow return}} \log d_C(f^{n_j}x) \geq \sum_{n_1 \leq n_j \leq n_t: \text{deep return}} \log d_C(f^{n_j}x).$$

### 3. INDUCING TO A LARGE SCALE

Let

$$(6) \quad M_0 := \left\lfloor \frac{2}{\lambda} \log(1/\delta) \right\rfloor = \left\lfloor \frac{2\alpha N_0}{\lambda} \right\rfloor,$$

where the square bracket denotes the integer part. Let  $|\cdot|$  denote the one-dimensional Lebesgue measure. In this section we prove

**Proposition 3.1.** *For an arbitrary interval  $I$  with  $\frac{\delta}{10} \leq |I| \leq \delta$ , there exists a countable partition  $\mathcal{P}$  of  $I$  into intervals and a stopping time function  $S: \mathcal{P} \rightarrow \{n \in \mathbb{N}: n \geq M_0\}$  such that:*

- (a) *for each  $\omega \in \mathcal{P}$ ,  $|f^{S(\omega)}\omega| \geq \sqrt{\delta}$  and  $|(f^{S(\omega)})'\omega| \geq 1/\delta^{\frac{1}{3}} > 1$ ;*
- (b) *the distortion of  $f^{S(\omega)}|_{\omega}$  is uniformly bounded. More precisely, for all  $x, y \in \omega$ ,*

$$\left| \frac{|(f^{S(\omega)})'x|}{|(f^{S(\omega)})'y|} - 1 \right| \leq \frac{1}{\sqrt{\delta}} |f^{S(\omega)}x - f^{S(\omega)}y|;$$

- (c)  $|\{S \geq n\}| \leq \delta^{\frac{11}{12}} L^{-\frac{\lambda n}{24}}$  *holds for every  $n > 0$ . Here,  $\{S \geq n\}$  is the union of all  $\omega \in \mathcal{P}$  such that  $S(\omega) \geq n$ .*

In Section 3.1 we define and describe the combinatorics of the partition  $\mathcal{P}$  and the stopping time  $S$ . (a) (b) follow from these definitions. In Section 3.2 we prove (c), assuming some key estimates on the measure of a set with a given combinatorics. In Section 3.3 we prove this key estimate.

**3.1. Combinatorial structure.** For each  $n \geq 0$ , considering  $n$ -iterates we construct a mod 0 partition  $\widehat{\mathcal{P}}_n$  of  $I$ . This construction is designed so that: each element of  $\mathcal{P}$  is an element of some  $\widehat{\mathcal{P}}_n$ ;  $\omega \in \mathcal{P} \cap \widehat{\mathcal{P}}_n$ , if and only if  $S(\omega) = n$ .

Let  $\widehat{\mathcal{P}}_0 = \{I\}$ , the trivial partition of  $I$ . Let  $n \geq 1$  and  $\omega \in \widehat{\mathcal{P}}_{n-1}$ . Then  $\widehat{\mathcal{P}}_n|_{\omega}$  is defined as follows:

*Case I:  $f^{n-1}\omega$  does not meet  $C \cup S$ .* We cut  $\omega$  from the left to the right, so that each subinterval has the form  $[x, x + D_n(x)]$ . If the rightmost interval does not have this form, then we take it together with the adjacent interval.

*Case II:  $f^{n-1}\omega$  meets  $C \cup S$ .* Consider a subinterval of  $\omega$  whose  $f^{n-1}$ -image does not meet  $C \cup S$  in its interior. Let  $\omega'$  denote any maximal interval with this property. We cut the right half of  $\omega'$  from the left to the right, as in Case I. We cut the left half of  $\omega'$  from the right to the left, analogously to Case I.

Let us record some basic properties of the partitions.

(P1) *Non-triviality.* For every  $n \geq M_0$ ,  $\widehat{\mathcal{P}}_n \neq \{I\}$  holds. Indeed, if  $I \cap C_\delta \neq \emptyset$ , then  $D_1(x) < d_S(x)/\sqrt{L} \ll \delta/10$  holds for  $x \in I \cap C_\delta$ , while  $|I| \geq \delta/10$  by the assumption. Hence  $I$  is subdivided in the construction of  $\widehat{\mathcal{P}}_1$ , that is  $I \notin \widehat{\mathcal{P}}_1$ . If  $I \cap C_\delta = \emptyset$ , then by Lemma 2.3,

either (i) there exists  $n \leq M_0$  such that  $f^i I \cap C_\delta = \emptyset$  for every  $0 \leq i < n$  and  $f^n I \cap C_\delta \neq \emptyset$ , or else (ii) there exists  $n \leq M_0$  such that  $f^n(\omega) = S^1$ , by the nature of the singularities. If (i) holds, then the same reasoning to the first case gives  $I \notin \widehat{\mathcal{P}}_{n+1}$ . If (ii) holds, then clearly  $I \notin \widehat{\mathcal{P}}_{n+1}$ .

(P2) *Bounded distortion.* By (P1), if  $n \geq M_0$ , then for each  $\omega \in \widehat{\mathcal{P}}_n$ ,  $D_n(x) \leq |\omega| \leq 10D_n(x)$  holds for some  $x \in \omega$ . From Lemma 2.2 (see Remark 2.1), the distortion of  $f^n|_\omega$  is bounded.

(P3) *Uniform expansion.* Let  $n \geq M_0$ ,  $\omega \in \widehat{\mathcal{P}}_n$  and suppose that  $|f^n \omega| \geq \sqrt{\delta}$ . From  $|\omega| \leq \delta$  and the second estimate in Lemma 2.2,

$$(7) \quad |(f^n)'x| \geq \frac{1}{2} \frac{|f^n \omega|}{|\omega|} \geq \frac{1}{2\sqrt{\delta}} > \frac{1}{\delta^{\frac{1}{3}}} \quad \forall x \in \omega.$$

**Definition 3.1.** Given  $\omega_n \in \widehat{\mathcal{P}}_n$  and  $k \in [0, n)$ , let  $\omega_k$  denote the unique element of  $\widehat{\mathcal{P}}_k$  which contains  $\omega_n$ . Let  $n \geq M_0$ . We say  $\omega_n \in \widehat{\mathcal{P}}_n$  reaches a large scale at time  $n$  if

$$n = \min \left\{ i \in [M_0, n] : |f^i \omega_i| \geq \sqrt{\delta} \right\}.$$

Let  $\mathcal{P}_n$  denote the collection of all elements of  $\widehat{\mathcal{P}}_n$  which reach a large scale at time  $n$ . Let  $\mathcal{P} = \bigcup_n \mathcal{P}_n$ . Define a stopping time function  $S: \mathcal{P} \rightarrow \mathbb{N}$  by  $S(\omega) = n$  for each  $\omega \in \mathcal{P}_n$ . Let  $\{S \geq n\}$  denote the union of all  $\omega \in \mathcal{P}$  such that  $S(\omega) \geq n$ . Let

$$\mathcal{P}'_n = \left\{ \omega_n \in \widehat{\mathcal{P}}_n : |f^i \omega_i| < \sqrt{\delta} \quad M_0 \leq \forall i \leq n \right\},$$

and let  $|P_n| = \sum_{\omega \in \mathcal{P}'_n} |\omega|$ . To show that  $\mathcal{P}$  is a mod 0 partition of  $I$ , it suffices to show  $|P_n| \rightarrow 0$  as  $n \rightarrow \infty$ . Then,  $|\{S \geq n\}| = |P_n|$  holds. (a) follows from (P3). (b) follows from the second estimate in Lemma 2.2.

**3.2. Exponential tails.** To prove (c), we have to show that  $|P_n|$  decays exponentially.

**Lemma 3.1.** *If  $n \geq M_0$ ,  $\omega \in \widehat{\mathcal{P}}_n$  and  $f^i(\omega) \cap C_\delta = \emptyset$  for every  $0 \leq i < n$ , then  $|f^n \omega| \geq \sqrt{\delta}$ .*

*Proof.* (P1) gives  $|\omega| \geq D_n(x)$  for some  $x \in \omega$ . (4) gives

$$|(f^n)'x|^{-1} |D_n(x)|^{-1} = \sqrt{L} \cdot \sum_{i=0}^{n-1} |(f^n)'x|^{-1} d_i(x)^{-1} \leq \frac{\sqrt{L}}{\delta^{\frac{1}{3}}}.$$

Taking reciprocals and then using the bounded distortion of  $f^n|_\omega$ , we obtain the inequality.  $\square$

To each  $\omega_n \in \mathcal{P}'_n$  we assign an *itinerary*

$$\mathbf{i} = (\nu_1, r_1, c_1), (\nu_2, r_2, c_2), \dots, (\nu_q, r_q, c_q)$$

which has the following interpretation. Let  $x_*$  denote the mid point of  $\omega_n$ . Then  $0 \leq \nu_1 < \dots < \nu_q < n$  are all the deep returns of the orbit of  $x_*$  before  $n$ ; for each  $i \in [1, q]$ ,  $f^{\nu_i} x_*$  is bound to  $c_i \in C$  and  $r_i$  is the unique integer such that  $|c_i - f^{\nu_i} x_*| \in (L^{-r_i}, L^{-r_i+1}]$ . Let  $P_n(\mathbf{i})$  denote the union of all elements of  $\mathcal{P}'_n$  with an itinerary  $\mathbf{i}$ . Lemma 3.1 gives  $|\{S \geq n\}| = \sum_{\mathbf{i}} |P_n(\mathbf{i})|$ , where the sum ranges over all feasible itineraries.

**Lemma 3.2.**  $|P_n(\mathbf{i})| < L^{-\frac{1}{3}R}$ , where  $R = r_1 + r_2 + \dots + r_q$ .



We finish the proof of (c) assuming the conclusion of this lemma. First, we count the number of all itineraries with the same  $R$  as follows. First, two consecutive returns to  $C_\delta$  are separated at least by  $\alpha N_0$ , and thus the largest possible number of returns in the first  $n$  iterates is  $n/\alpha N_0$ . Second, given  $q \in [1, n/\alpha N_0]$ , there are at most  $\binom{n}{q}$  number of ways to choose the positions of  $q$  number of free returns in  $[0, n]$ . For each such way  $(n_1, \dots, n_q)$  there is at most  $\binom{R+q}{q}$  number of ways to assign  $r_1, \dots, r_q$  with  $r_1 + \dots + r_q = R$ . Hence

$$(8) \quad |\{S \geq n\}| = \sum_R \sum_{\substack{P_n(\mathbf{i}) \\ r_1 + \dots + r_q = R}} |P_n(\mathbf{i})| = \sum_R \sum_{q=1}^{R/\alpha N_0} \binom{n}{q} \binom{R+q}{q} L^{-\frac{R}{3}} \leq \sum_R L^{-\frac{R}{4}}.$$

The last inequality follows from Stirling's formula for factorials.

To get a lower bound on  $R$ , take one element  $\omega \in \mathcal{P}'_n$  with an itinerary  $\mathbf{i}$  and let  $0 \leq n_1 < \dots < n_t < n$  denote all the free (both shallow and deep) returns of the mid point  $x_*$  of  $\omega$  before  $n$ . Let  $p_k$  denote the bound period for  $n_k$  and  $s_k$  the unique integer such that  $d_C(f^{n_k} x_*) \in (L^{-s_k}, L^{-s_k+1}]$  holds.

**Lemma 3.3.** *For every  $1 \leq k < t$ ,  $n_{k+1} - n_k \leq \frac{3s_k}{\lambda}$ .*

*Proof.* We assume  $n_{k+1} > n_k + \frac{3s_k}{\lambda}$  and derive a contradiction. By the upper estimate of the bound period in Lemma 2.4,  $n_{k+1} > n_k + p_k + \frac{s_k}{\lambda}$  holds. By Lemma 2.3,

$$|(f^{n_{k+1}-n_k-p_k})' f^{n_k+p_k} x_*| \geq L^{s_k} \geq |d_C(f^{n_k} x_*)|^{-1}.$$

For every  $1 \leq j \leq k$ ,

$$\begin{aligned} |(f^{n_{k+1}-n_j-p_j})' f^{n_j+p_j} x_*| &= |(f^{n_{k+1}-n_k-p_k})' f^{n_k+p_k} x_*| |(f^{n_k+p_k-n_j-p_j})' f^{n_j+p_j} x_*| \\ &\geq |d_C(f^{n_k} x_*)|^{-1} L^{\frac{\lambda}{3}(n_k+p_k-n_j-p_j)}, \end{aligned}$$

and therefore

$$\begin{aligned} \sum_{i=n_j}^{n_j+p_j} |(f^{n_{k+1}})' x_*|^{-1} d_i^{-1}(x_*) &= |(f^{n_{k+1}-n_j-p_j})' f^{n_j+p_j} x_*|^{-1} \cdot \sum_{i=n_j}^{n_j+p_j} |(f^{n_j+p_j})' x_*|^{-1} d_i^{-1}(x_*) \\ &\leq L^{-\frac{\lambda}{3}(n_k+p_k-n_j-p_j)} |d_C(f^{n_k} x_*)|^{1-\frac{18\alpha}{\lambda}} \leq L^{-\frac{\lambda}{3}(n_k+p_k-n_j-p_j)}. \end{aligned}$$

For the second factor in the right-hand-side of the equality we have used (3). Summing this over all  $1 \leq j \leq k$  and adding the contribution from all the free iterates outside of  $C_\delta$  which was estimated in (4),

$$\begin{aligned} \sum_{i=0}^{n_{k+1}-1} |(f^{n_{k+1}})' x_*|^{-1} d_i^{-1}(x_*) &= \sum_{j=1}^k \sum_{i=n_j}^{n_j+p_j} + \sum_{i \in \cup_{j=1}^k (n_j+p_j, n_{j+1})} \\ &\leq \frac{1}{\delta^{\frac{1}{3}}} + \sum_{j=1}^k L^{-\frac{\lambda}{3}(n_k+p_k-n_j-p_j)} \leq \frac{1}{\delta^{\frac{1}{3}}} + \sum_{i=0}^{\infty} L^{-\frac{\lambda i}{3}} < \frac{2}{\delta^{\frac{1}{3}}}. \end{aligned}$$

Taking reciprocal and then using the bounded distortion of  $f^{n_{k+1}}|_{\omega_{n_{k+1}}}$ , we have  $|f^{n_{k+1}} \omega_{n_{k+1}}| \geq \sqrt{\delta}$ . This yields a contradiction to the assumption  $S(\omega) \geq n$ .  $\square$

Summing the inequality in Lemma 3.3 over all  $1 \leq k < t$  gives

$$(9) \quad n_t \leq n_1 + \frac{3}{\lambda} \sum_{k=1}^{t-1} s_k.$$

From this point on we assume  $n \geq 2M_0$ . Then  $n_1 \leq n/2$  holds, for otherwise  $n_1 > n/2 \geq M_0$  and  $|f^{n_1}\omega_{n_1}| \geq \sqrt{\delta}$  would follow from Lemma 3.1, a contradiction to  $S(\omega) \geq n$ . We have

$$n_t + p_t \leq \frac{n}{2} + p_t + \frac{3}{\lambda} \sum_{k=1}^{t-1} s_k \leq \frac{n}{2} + \frac{3}{\lambda} \sum_{k=1}^t s_k \leq \frac{n}{2} + \frac{6}{\lambda} R.$$

We have used  $p_t \leq \frac{2}{\lambda} s_t$  for the second inequality which follows from Lemma 2.4, and Lemma 2.6 for the last. When  $n$  is bound, then  $n < n_t + p_t$  holds, and thus the above inequality yields  $R \geq \frac{\lambda n}{12}$ . When  $n$  is free, repeating the argument in the proof of Lemma 3.3 we get  $n - n_t \leq \frac{3s_t}{\lambda}$ . Combining this with (9) yields the same lower bound of  $R$ . Consequently we obtain  $|\{S \geq n\}| \leq L^{-\frac{\lambda n}{24}}$  for every  $n \geq 2M_0$ . As  $|I| \leq \delta$ ,  $|\{S \geq n\}| \leq \delta L^{\frac{2\lambda M_0}{24}} L^{-\frac{\lambda n}{24}}$  holds for every  $n > 0$ . The choice of  $M_0$  in (6) gives  $L^{\frac{\lambda M_0}{12}} \leq \delta^{-\frac{1}{12}}$ , and the desired inequality holds.

**3.3. Proof of Lemma 3.2.** We first treat the case  $\nu_1 > 0$ . For all  $x \in P_n(\mathbf{i})$  and each  $k \in [1, q]$  we define an interval  $I_k(x)$  in such a way that  $f^{\nu_k}$  sends  $I_k(x)$  to an interval injectively with bounded distortion. Let  $\hat{I}_k(x)$  denote the interval of length  $D_{\nu_k}(x)$  centered at  $x$ . By Lemma 2.2, the distortion of  $f^{\nu_k}|_{\hat{I}_k(x)}$  is uniformly bounded, while  $f^{\nu_k}$  may not be injective on  $\hat{I}_k(x)$ . If  $|f^{\nu_k}(\hat{I}_k(x))| \leq 1/4$ , define  $I_k(x) = \hat{I}_k(x)$ . Otherwise, define  $I_k(x)$  to be the interval of length  $|\hat{I}_k(x)|/(10|f^{\nu_k}(\hat{I}_k(x))|)$  centered at  $x$ . By construction,  $f^{\nu_k}$  is injective on  $I_k(x)$ .

*Notation.* For a compact interval  $I$  centered at  $x$  and  $r > 0$ , let  $r \cdot I$  denote the interval of length  $r|I|$  centered at  $x$ .

For each  $k \in [1, q]$ , we choose a subset (possibly infinite)  $\{x_{k,i}\}_i$  of  $P_n(\mathbf{i})$  with the following properties:

- (i) the intervals  $\{I_k(x_{k,i})\}_i$  are pairwise disjoint and  $P_n(\mathbf{i}) \subset \bigcup_i L^{-r_k/3} \cdot I_k(x_{k,i})$ ;
- (ii) for each  $k \in [2, q]$  and  $x_{k,i}$  there exists  $x_{k-1,j}$  such that  $I_k(x_{k,i}) \subset 2L^{-r_{k-1}/3} \cdot I_{k-1}(x_{k-1,j})$ .

Let  $M_k = \sum_i |I_k(x_{k,i})|$ . It follows that  $|P_n(\mathbf{i})| \leq L^{-r_q/3} M_q$  and  $M_k \leq L^{-r_{k-1}/3} M_{k-1}$ , and therefore  $|P_n(\mathbf{i})| \leq L^{-\frac{1}{3} \sum_{k=1}^q r_k}$ , and the desired estimate holds.

For the definition of the subsets we need two combinatorial lemmas. The following elementary fact from Lemma 2.5 is used in the proofs of these two lemmas:  $(f^{\nu_k}|_{I_k(x)})^{-1}(c_k)$  consists of a single point and  $(f^{\nu_k}|_{I_k(x)})^{-1}(c_k) \subset L^{-r_k/3} \cdot I_k(x)$ .

**Lemma 3.4.** *If  $x, y \in P_n(\mathbf{i})$  and  $y \notin I_k(x)$ , then  $I_k(x) \cap I_k(y) = \emptyset$ .*

*Proof.* Suppose  $I_k(x) \cap I_k(y) \neq \emptyset$ . Lemma 2.2 gives  $|I_k(x)| \approx |I_k(y)|$ . This and  $y \in I_k(x)$  imply  $(f^{\nu_k}|_{I_k(x)})^{-1}(c_k) \neq (f^{\nu_k}|_{I_k(y)})^{-1}(c_k)$ . On the other hand, by the definition of the intervals  $I_k(\cdot)$ ,  $f^{\nu_k}$  is injective on  $I_k(x) \cup I_k(y)$ . A contradiction arises.  $\square$

**Lemma 3.5.** *If  $x, y \in P_n(\mathbf{i})$  and  $y \in L^{-r_k/3} \cdot I_k(x)$ , then  $I_{k+1}(y) \subset 2L^{-r_k/3} \cdot I_k(x)$ .*

*Proof.* We have  $(f^{\nu_k}|_{I_k(x)})^{-1}(c_k) \notin I_{k+1}(y)$ , for otherwise the distortion of  $f^{\nu_{k+1}}|_{I_{k+1}(y)}$  is unbounded. This and the assumption together imply that one of the connected components of  $I_{k+1}(y) - \{y\}$  is contained in  $L^{-r_k/3} \cdot I_k(x)$ . This implies the inclusion.  $\square$

We are in position to choose subsets  $\{x_{k,i}\}_i$  satisfying (i) (ii). Lemma 3.4 with  $k = 1$  allows us to pick a subset  $\{x_{1,i}\}$  such that the corresponding intervals  $\{I_1(x_{1,i})\}$  are pairwise disjoint, and altogether cover  $P_n(\mathbf{i})$ . Indeed, pick an arbitrary  $x_{1,1}$ . If  $I_1(x_{1,1})$  covers  $P_n(\mathbf{i})$ , then the claim holds. Otherwise, pick  $x_{1,2} \in P_n(\mathbf{i}) - I_1(x_{1,1})$ . By Lemma 3.4,  $I_1(x_{1,1}), I_1(x_{1,2})$  are disjoint. Repeat this. By Lemma 3.4, we end up with pairwise disjoint intervals. To check the inclusion in (i), let  $x \in I_1(x_{1,i}) - L^{-r_1/3} \cdot I_1(x_{1,i})$ . By Lemma 2.5,  $|f^{\nu_1}x - c_1| \gg L^{-r_k}$  holds. Hence  $x \notin P_n(\mathbf{i})$ .

Given  $\{x_{k-1,j}\}_j$ , we choose  $\{x_{k,i}\}_i$  as follows. For each  $x_{k-1,j}$ , similarly to the previous paragraph it is possible to choose parameters  $\{x_m\}_m$  in  $P_n(\mathbf{i}) \cap L^{-r_{k-1}/3} \cdot I_{k-1}(x_{k-1,j})$  such that the corresponding intervals  $\{I_k(x_m)\}_m$  are pairwise disjoint and altogether cover  $P_n(\mathbf{i}) \cap L^{-r_{k-1}/3} \cdot I_{k-1}(x_{k-1,j})$ . In addition, Lemma 3.5 gives  $\bigcup_m I_k(x_m) \subset 2L^{-r_{k-1}/3} \cdot I_{k-1}(x_{k-1,j})$ . Let  $\{x_{k,i}\}_i = \bigcup_j \{x_m\}$ .

It is left to treat the case  $\nu_1 = 0$ . In this particular case, by definition of  $\mathbf{i}$ ,  $P_n(\mathbf{i})$  is contained in  $(-L^{-r_1+1}, L^{-r_1+1})$ . Hence, the desired estimate holds if  $q = 1$ . If  $q > 1$ , then in the same way as above, it is possible to show  $|P_n(\mathbf{i})| \leq L^{-\frac{1}{3}(R-r_1)} 2L^{-r_1+1}$ , which is  $\leq L^{-\frac{R}{3}}$ . This finishes the proof of Lemma 3.2.  $\square$

#### 4. INDUCED MARKOV MAP ON $S^1$

In this section we construct an induced Markov map on  $S^1$  and complete the proofs of the theorems.

**Proposition 4.1.** *There exist a partition  $\mathcal{Q}$  of a full measure set of  $S^1$  into a countable number of open intervals and a return time function  $R: \mathcal{Q} \rightarrow \{n \in \mathbb{N}: n > M_0\}$  with the following properties. For each  $\omega \in \mathcal{Q}$ ,  $F := f^R$  sends  $\omega$  injectively, so that  $\overline{F(\omega)} = S^1$ . There exists  $K > 0$  such that for all  $\omega \in \mathcal{Q}$  and all  $x, y \in \omega$ ,*

$$(10) \quad \left| \frac{|F'(x)|}{|F'(y)|} - 1 \right| \leq K|F(x) - F(y)|.$$

*In addition,  $|\{R = n\}| \leq \delta L^{-\frac{\lambda n}{26}}$  holds for every  $n > M_0$ . Here,  $\{R = n\}$  denotes the union of  $\omega \in \mathcal{Q}$  such that  $R(\omega) = n$ .*

Our inducing time consists of four explicit parts: the first part is used to recover from the small derivatives near the critical set (Proposition 2.4); in the second, intervals reach a “large scale scale” (Proposition 3.1) and in the third they reach a neighborhood of the critical set. The last part is used to completely “wrap” the circle.

In Section 4.1 we prove a key lemma used in the third and fourth parts of the inducing time. In Section 4.2 we construct the induced map  $F$  with the desired properties. In Section 4.3 we prove Theorem A. In Section 4.4 we prove Theorem B.

**4.1. Inducing to the entire  $S^1$ .** We show that intervals with scale  $\sqrt{\delta}$  soon grow to the entire  $S^1$ . There are two scenarios for this growth. One is to take advantage of the nature of the singularities. The other is to follow the initial iterates of the critical orbits, which are kept out of  $C_\sigma, S_\sigma$  by the standing hypothesis (a) in Sect.2.4.

We first show that intervals with scale  $\sqrt{\delta}$  soon reach critical or singular neighborhoods.

**Lemma 4.1.** *For any interval  $\omega$  of length  $\geq \sqrt{\delta}/3$ , there exist a subinterval  $\omega'$  and an integer  $M \leq M_0$  such that  $d_C(f^i\omega') \geq \delta$ ,  $d_S(f^i\omega') \geq \delta$  for every  $0 \leq i < M$  and  $f^M(\omega')$  coincides with one component of  $C_\delta \cup S_\delta$ .*

*Proof.* We iterate  $\omega$ , deleting all parts that fall into  $C_\delta \cup S_\delta$ . Suppose that this is continued up to step  $n$ , and that for every  $i \leq n$ , none of these deleted segments is  $< 2\delta$  in length. By the assumption, the number of deleted segments at step  $i \leq n$  is  $\leq 2$ . By Lemma 2.3, all deleted parts in  $\omega$  are  $\leq 4\delta \sum_{i=0}^n L^{-2\lambda_i}$  in length. Hence, the undeleted segment in  $f^n\omega$  is  $\geq \left(\sqrt{\delta}/3 - 4\delta \sum_{i=0}^n L^{-\lambda_i}\right) \delta L^{2\lambda_n}$  in length. It follows that before step  $M_0$  there must come a point when our claim is fulfilled.  $\square$

For convenience, let us introduce the following language.

**Definition 4.1.** Let  $\varepsilon > 0$  and  $M > 0$  an integer. A pair of open intervals  $(\omega, \tilde{\omega})$  with  $\tilde{\omega} \subset \omega$  is a *good  $(\varepsilon, M)$ -pair* if: (i)  $|\tilde{\omega}| \geq \varepsilon|\omega|$ ; (ii)  $\omega \setminus \tilde{\omega}$  has two components and their lengths are  $\geq \sqrt{\delta}/3$ ; (iii)  $f^M$  is injective  $\tilde{\omega}$  and  $\overline{f^M(\tilde{\omega})} = S^1$ ; (iv)  $d_C(f^i\tilde{\omega}) \geq \varepsilon$ ,  $d_S(f^i\tilde{\omega}) \geq \varepsilon$  for every  $0 \leq i < M$ .

**Lemma 4.2.** *There exists  $0 < \varepsilon_0 < 1$  such that for any interval  $\omega$  of length  $\geq \sqrt{\delta}$ , there exist a subinterval  $\tilde{\omega}$  in its middle third and an integer  $k \leq 2M_0$  such that  $(\omega, \tilde{\omega})$  is a good  $(\varepsilon_0, k)$ -pair.*

*Proof.* Take a subinterval  $\omega'$  in the middle third of  $\omega$  and an integer  $M$  for which the conclusion of Lemma 4.1 holds. We deal with two cases separately.

*Case I:*  $f^M\omega' \subset S_\delta$ . By the nature of the singularity, there exists a subinterval  $\omega'' \subset f^M\omega'$  such that  $d_S(\omega'') \geq \delta/10$ ,  $f|_{\omega''}$  is injective and  $\overline{f(\omega'')} = S^1$ . Let  $\tilde{\omega} = f^{-M}(\omega'')$  and  $k = M + 1$ .

*Case II:*  $f^M\omega' \subset C_\delta$ . Let  $N_1 = [10\alpha N_0]$ . Let  $c$  denote the critical point in  $f^M\omega'$ . By the definition of  $\delta$ ,  $f^M\omega'$  contains  $I_{N_1}(c)$ .

**Sublemma 4.1.**  $f^{N_1+1}(I_{N_1}(c)) = S^1$ , and  $d_C(f^i I_{N_1}(c)) \geq \sigma/2$ ,  $d_S(f^i I_{N_1}(c)) \geq \sigma/2$  for every  $1 \leq i \leq N_1$ .

We finish the proof of Lemma 4.2 assuming the conclusion of this sublemma. Take a subinterval  $J \subset I_{N_1}(c)$  on which  $f^{N_1+1}$  is injective and  $\overline{f^{N_1+1}(J)} = S^1$  holds. Let  $\tilde{\omega} = f^{-M}(J)$ , and  $k = M + N_1 + 1$ . Sublemma 4.1 gives  $d_C(f^i\tilde{\omega}) \geq \sigma/2$ ,  $d_S(f^i\tilde{\omega}) \geq \sigma/2$  for every  $M < i < k$ . As  $f^M(\tilde{\omega}) \subset I_{N_1}(c)$ ,  $d_C(f^M\tilde{\omega}) \geq \sqrt{D_{N_1+1}(c)/(K_0L)}$  holds. (6) gives  $k \leq 2M_0$ .

In either of the two cases, the  $M$ -iterates of  $\omega'$  are kept out of  $C_\delta \cup S_\delta$ . Hence, the distortion of  $f^M|_{\omega'}$  is uniformly bounded and there exists a uniform constant  $0 < \varepsilon'_0 < 1$  such that  $|\tilde{\omega}| \geq \varepsilon'_0|\omega'|$ . Set

$$\varepsilon_0 = \min \left( \delta/10, \sigma/2, \inf_{c \in C} \sqrt{D_{N_1+1}(c)/(K_0L)}, \varepsilon'_0 \right).$$

Then,  $(\omega, \tilde{\omega})$  is a good  $(\varepsilon_0, k)$ -pair.

It is left to prove Sublemma 4.1. Let  $f^{i+1}(c) = v_i$ . The next standing hypothesis (a) in Sect.2.4 is used:  $d_C(v_i) \geq \sigma$ ,  $d_S(v_i) \geq \sigma$  for every  $0 \leq i < N_0$ .

Let  $l = \min\{|x - y| : x \in C, y \in S\} > 0$ . It is easy to see that,  $d_C(v_i) \geq \sigma$ ,  $d_S(v_i) \geq \sigma$  give

$$d_C(v_i)d_S(v_i) \geq \sigma(l - \sigma) \geq l\sigma/2,$$

where the last inequality holds for sufficiently large  $L$ . In view of (2) and  $|(f^{N_1})'(v_0)| \geq (L\sigma/K_0)^{N_1-i}|(f^i)'(v_0)|$ ,

$$\frac{1}{\sqrt{L}} \frac{d_{N_1}(v_0)}{D_{N_1}(v_0)} = \sum_{i=0}^{N_1-1} \frac{|(f^i)'(v_0)|}{|(f^{N_1})'(v_0)|} \frac{d_C(v_{N_1})d_S(v_{N_1})}{d_C(v_i)d_S(v_i)} \leq \sum_{i=0}^{N_1-1} \frac{2}{(L\sigma/K_0)^{N_1-i}\sigma l} \leq \frac{1}{\sqrt{L}}.$$

This yields  $D_{N_1}(v_0)/D_{N_1+1}(v_0) = 1 + \sqrt{L}D_{N_1}(v_0)d_{N_1}^{-1}(v_0) \geq 2$ . We also have

$$|(f^{N_1})'(v_0)D_{N_1}(v_0)|^{-1} = \sqrt{L} \sum_{i=0}^{N_1-1} \frac{|(f^i)'(v_0)|}{|(f^{N_1})'(v_0)|} \frac{1}{d_C(v_i)d_S(v_i)} \leq \frac{2}{\sqrt{L}\sigma^2 l(1 - 1/(L\sigma/K_0))} \leq L^{-\frac{1}{7}},$$

where the last inequality follows from  $\sigma = L^{-\frac{1}{6}}$ . Hence

$$\begin{aligned} |f^{N_1+1}(I_{N_1}(c))| &\geq \frac{1}{2}|(f^{N_1})'v_0||f(I_{N_1}(c))| \geq \frac{1}{2}|(f^{N_1})'v_0|(D_{N_1}(v_0) - D_{N_1+1}(v_0)) \\ &\geq \frac{1}{4}|(f^{N_1})'D_{N_1}(v_0)| \geq L^{\frac{1}{8}} \gg 1. \end{aligned}$$

Hence, the first claim holds. The second follows from the standing hypothesis and  $|f^i(I_{N_1}(c))| \leq |(f^i)'v_0|D_{N_1}(v_0) \leq \frac{1}{\sqrt{L}}|(f^i)'v_0|d_i(v_0) \leq \frac{1}{\sqrt{L}}$ . This completes the proof of Sublemma 4.1.  $\square$

**4.2. Full return map.** We now define a partition  $\mathcal{Q}$  of  $S^1$  and a return time function  $R: \mathcal{Q} \rightarrow \mathbb{N}$ . First of all, cut  $S^1$  into pairwise disjoint intervals of lengths from  $\delta/10$  to  $\delta$ . For each interval, consider its partition  $\mathcal{P}$  and the associated return time function  $S: \mathcal{P} \rightarrow \mathbb{N}$ , given by Proposition 3.1. By Lemma 4.2, for each  $\omega_1 \in \mathcal{P}$  there exists a subinterval  $\tilde{\omega}_1$  and an integer  $M_1$  such that  $(f^{S(\omega_1)}\omega_1, f^{S(\omega_1)}\tilde{\omega}_1)$  is a good  $(\varepsilon_0, M_1)$ -pair. Let  $\tilde{\omega}_1 \in \mathcal{Q}$  and  $R(\tilde{\omega}_1) := S(\omega_1) + M_1$ . Each component of  $\omega_1 \setminus \tilde{\omega}_1$  is said to *have 1 large scale times*.

Subdivide each component of  $f^{S(\omega_1)}(\omega_1) \setminus f^{S(\omega_1)}(\tilde{\omega}_1)$ , which is of length  $\geq \sqrt{\delta}/3$  by the definition of good pairs, into intervals of lengths from  $\delta/10$  to  $\delta$ . To each interval, consider again its partition  $\mathcal{P}$  and the stopping time function  $S$  given by Proposition 3.1. For each element  $\omega_2$  of the partition, there exist an integer  $M_2$  and a subinterval  $\tilde{\omega}_2$  such that  $(f^{S(\omega_2)}\omega_2, f^{S(\omega_2)}\tilde{\omega}_2)$  is a good  $(\varepsilon_0, M_2)$ -pair. Let  $f^{-S(\omega_1)}(\tilde{\omega}_2) \in \mathcal{Q}$  and  $R(\tilde{\omega}_2) := S(\omega_1) + S(\omega_2) + M_2$ . Each component of  $f^{-S(\omega_1)}\omega_2 \setminus f^{-S(\omega_1)}\tilde{\omega}_2$  is said to *have 2 large scale times*, and so on.

**4.2.1. Bounded distortion.** We verify (10). By construction, for each  $\omega \in \mathcal{Q}$  there exists an associated sequence of large scale times

$$0 = S_0 < S_1 < S_2 < \dots < S_{q(\omega)} < R(\omega)$$

with  $R(\omega) = S_{q(\omega)} + t(\omega)$  and  $t(\omega) \leq 2M_0$ . For each  $0 \leq i < q$ ,  $|(f^{S_{i+1}-S_i})'| \geq 1/\delta^{\frac{1}{3}}$  holds on  $f^{S_i}\omega$ . The second estimate in Lemma 2.2 gives

$$(11) \quad \ln \frac{|(f^{S_{i+1}-S_i})'(f^{S_i}x)|}{|(f^{S_{i+1}-S_i})'(f^{S_i}y)|} \leq \frac{1}{\sqrt{\delta}} |f^{S_{i+1}}(x) - f^{S_{i+1}}(y)| \leq \frac{\delta^{\frac{q-i-1}{3}}}{\sqrt{\delta}} |f^{S_q}(x) - f^{S_q}(y)|.$$

The additional at most  $2M_0$  iterates after the last large scale time  $S_q$  does not significantly affect the distortion. Consequently, (10) holds.

4.2.2. *Exponential tails.* For each  $1 \leq i < n$ , let  $\mathcal{Q}_n^{(i)}$  denote the collection of all  $\omega \in \mathcal{Q}$  which have exactly  $i$  large scale times before  $n$  and  $R(\omega) = n$ . Let  $|\mathcal{Q}_n^{(i)}| = \sum_{\omega \in \mathcal{Q}_n^{(i)}} |\omega|$ . By construction, two consecutive large scale times are separated at least by  $M_0$ , and  $|\{R = n\}| = \sum_{1 \leq i \leq n/M_0} |\mathcal{Q}_n^{(i)}|$  holds.

We estimate the measure of  $\mathcal{Q}_n^{(i)}$ . For  $2 \leq i \leq n/M_0$  and an  $i$  string  $(k_1, \dots, k_i)$  of positive integers with  $k_1 + \dots + k_i < n$ , let

$$\mathcal{Q}_n(k_1, \dots, k_i) = \{\omega \in \mathcal{Q}_n^{(i)} : S_j - S_{j-1} = k_j \quad 1 \leq j \leq i\}.$$

Let  $|\mathcal{Q}_n(k_1, \dots, k_i)| = \sum_{\omega \in \mathcal{Q}_n(k_1, \dots, k_i)} |\omega|$ . For each  $\omega \in \mathcal{Q}_n(k_1, \dots, k_{i-1})$ , let

$$\mathcal{Q}_n(\omega, k_i) = \{\omega' \in \mathcal{Q}_n(k_1, \dots, k_i) : \omega' \subset \omega\}.$$

By definition,

$$|\mathcal{Q}_n(k_1, \dots, k_i)| = \sum_{\omega \in \mathcal{Q}_n(k_1, \dots, k_{i-1})} |\omega| \sum_{\omega' \in \mathcal{Q}_n(\omega, k_i)} \frac{|\omega'|}{|\omega|}.$$

To estimate the fraction, let  $\omega \in \mathcal{Q}_n(k_1, \dots, k_{i-1})$ . Proposition 3.1 gives

$$\sum_{\omega' \in \mathcal{Q}_n(\omega, k_i)} |f^{k_1 + \dots + k_{i-1}}(\omega')| \leq \delta^{\frac{11}{12}} L^{-\frac{\lambda k_i}{24}} |f^{k_1 + \dots + k_{i-1}}(\omega)|.$$

By construction,  $|f^{k_1 + \dots + k_{i-1}}(\omega)| \leq \delta$  holds. By (11), the distortion of  $f^{k_1 + \dots + k_{i-1}}|_{\omega}$  is uniformly bounded and

$$\sum_{\omega' \in \mathcal{Q}_n(\omega, k_i)} \frac{|\omega'|}{|\omega|} \leq 2\delta^{\frac{11}{12}} L^{-\frac{\lambda k_i}{24}}.$$

Hence we obtain  $|\mathcal{Q}_n(k_1, \dots, k_i)| \leq L^{-\frac{k_i}{24}} |\mathcal{Q}_n(k_1, \dots, k_{i-1})|$ . Using this inductively and then  $|\mathcal{Q}_n(k_1)| \leq \delta^{\frac{11}{12}} L^{-\frac{\lambda k_1}{24}}$  which follows from Proposition 3.1,

$$|\mathcal{Q}_n(k_1, \dots, k_i)| \leq \delta^{\frac{11}{12}} L^{-\frac{1}{24}(k_1 + \dots + k_i)}.$$

For any given  $m \in [n - 2M_0, n)$  and  $1 \leq i \leq n/M_0$ , the number of all feasible  $(k_1, \dots, k_i)$  with  $k_1 + \dots + k_i = m$  equals the number of ways of dividing  $m$  objects into  $i$  groups, which is  $\binom{m+i}{i}$ , and by Stirling's formula for factorials, this number is  $\leq e^{\beta m}$ , where  $\beta \rightarrow 0$  as  $L \rightarrow \infty$ . Hence we obtain

$$\begin{aligned} |\mathcal{Q}_n^{(i)}| &= \sum_{m=n-2M_0}^{n-1} \sum_{k_1 + \dots + k_i = m} |\mathcal{Q}_n(k_1, \dots, k_i)| \\ &\leq \delta^{\frac{11}{12}} \sum_{m=n-2M_0}^{n-1} L^{-\frac{m}{24}} \#\left\{(k_1, \dots, k_i) : \sum_{j=1}^i k_j = m\right\} \leq \delta^{\frac{11}{12}} L^{-\frac{\lambda}{25}(n-2M_0)}. \end{aligned}$$

The same inequality remains to hold for  $i = 1$ . Summing these over all  $1 \leq i \leq n/M_0$ ,

$$|\{R = n\}| = \sum_{1 \leq i \leq n/M_0} |\mathcal{Q}_n^{(i)}| \leq \delta^{\frac{11}{12}} L^{\frac{2\lambda M_0}{25}} L^{-\frac{\lambda n}{26}} \leq \delta L^{-\frac{\lambda n}{26}}.$$

The last inequality follows from  $L^{\frac{\lambda M_0}{12}} \leq \delta^{\frac{1}{6}}$ . This finishes the proof of (c).

**4.3. Proof of Theorem A.** We have constructed an induced Markov map with exponential tails and Lipschitz bounded distortion property. Then all the statements of Theorem A follow from the abstract scheme in [27]. See [[4] pp.644] for a concise explanation.

In fact, to apply [27], the right hand side (10) has to be bounded by a uniform constant multiplied by  $\beta^{s(x,y)}$ , where  $0 < \beta < 1$  and  $s(x,y)$  is a *separation time* [27]. This is a direct consequence of (10) and the uniform expansion of  $F$  on each  $\omega \in \mathcal{Q}$ .

**4.4. Proof of Theorem B.** Let  $\phi: S^1 \rightarrow \mathbb{R}$  be a Hölder continuous function which is coboundary. Let  $\psi \in L^2(\mu)$  satisfy  $\phi = \psi \circ f - \psi$ . We show that  $\psi$  has a version which is (Hölder) continuous<sup>1</sup> on the entire  $S^1$ .

For each  $n \geq 1$ , let  $\mathcal{H}_n$  denote the collection of inverse branches of  $F^n$ . Let  $\mathcal{F}_n$  denote the  $\sigma$ -algebra generated by the intervals  $h(S^1)$  for  $h \in \mathcal{H}_n$ . It is an increasing sequence of  $\sigma$ -algebras. For almost every  $x \in S^1$ , there exists an well-defined sequence  $\bar{h} = (h_1, h_2, \dots) \in \mathcal{H}_1^{\mathbb{N}}$  such that the element  $F_n(x)$  of  $\mathcal{F}_n$  containing  $x$  is given by  $F_n(x) = h_1 \circ \dots \circ h_n(S^1)$ . Equivalently,  $h_n$  is the unique element of  $\mathcal{H}_1$  such that  $F^{n-1}(x) \in h_n(S^1)$ .

The Martingale convergence theorem shows that, for almost every  $x \in S^1$  and for all  $\epsilon > 0$ ,

$$(12) \quad \frac{\text{Leb}\{x' \in F_n(x) : |\psi(x') - \psi(x)| > \epsilon\}}{\text{Leb}(F_n(x))} \rightarrow 0 \quad a.e. \ x \in S^1.$$

Take a point  $x_0$  such that this convergence holds. Let  $\bar{h} = (h_1, h_2, \dots)$  denote the corresponding sequence of  $\mathcal{H}$  and write  $\bar{h}_n = h_1 \circ \dots \circ h_n$ , so that  $F_n(x_0) = \bar{h}_n(S^1)$ . (12) and the bounded distortion of  $\bar{h}_n$  give, for all  $\epsilon > 0$ ,

$$\text{Leb}\{x \in S^1 : |\psi(\bar{h}_n x) - \psi(x_0)| > \epsilon\} \rightarrow 0.$$

Choose a subsequence  $(n_k)$  such that for all  $\epsilon > 0$ ,

$$\sum_{k=1}^{\infty} \text{Leb}\{x \in S^1 : |\psi(\bar{h}_{n_k} x) - \psi(x_0)| > \epsilon\} < \infty.$$

By the Borel-Cantelli lemma,  $\psi(\bar{h}_{n_k} x) \rightarrow \psi(x_0)$  holds for almost every  $x$ .

Let  $S_k(x) = \sum_{i=1}^{n_k} \phi(\bar{h}_i x)$ . As  $\psi(x) = \psi(\bar{h}_{n_k} x) + S_{n_k}(x)$ ,  $S_k(x)$  converges for almost every  $x$ . For all  $x$  such that this convergence holds, let  $S(x) = \lim_{k \rightarrow \infty} S_k(x)$ . The uniform contraction over all inverse branches and the Hölder continuity of  $\phi$  give  $|S_k(x) - S_k(y)| \leq K|x - y|^\eta$ , where  $\eta$  is the Hölder exponent of  $\phi$ . Passing to the limit we obtain  $|S(x) - S(y)| \leq K|x - y|^\eta$ , that is,  $S$  is continuous. As  $\psi(x) = \psi(x_0) + S(x)$ , it follows that  $\psi$  has a version which is continuous on the entire  $S^1$ .

Assume that  $\psi$  is not a constant function. Fix  $z, z'$  such that  $\psi(z) \neq \psi(z')$ . Fix a singular point  $y \in S$ . We evaluate the cohomologous equation along a sequence  $(x_n)$  with  $x_n \rightarrow y$ . By continuity,  $\phi(x_n) + \psi(x_n) \rightarrow \phi(y) + \psi(y)$  holds. To obtain a contradiction, it suffices to choose two sequences  $x_n \rightarrow y$ ,  $x'_n \rightarrow y$  so that  $\psi(fx_n), \psi(fx'_n)$  converge to different limits. By the nature of the singularities, for any sufficiently large  $n > 0$  there exists an interval  $I_n$  in  $[y, y + 1/n]$  such that  $f(I_n) = S^1$ . Pick two points  $x_n \in f^{-1}(z) \cap I_n$ ,  $x'_n \in f^{-1}(z') \cap I_n$ . Clearly,  $x_n \rightarrow y, x'_n \rightarrow y$  and  $\psi(fx_n) \rightarrow \psi(z), \psi(fx'_n) \rightarrow \psi(z')$  hold. This completes the proof of Theorem B.  $\square$

For later use in the next section, we prove

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<sup>1</sup>The same conclusion follows from Livšic regularity results [4, 7].

**Corollary 4.1.**  $\log |f'|$  is  $\mu$ -integrable and  $\bar{\lambda} := \int \log |f'| d\mu > 0$ .

*Proof.* Let  $\nu$  denote the acim of  $F$ . By the classical theorem, the density of  $\nu$  is uniformly bounded from above and below. From the uniform expansion and the bounded distortion of  $F$ , there exist  $K > 0$ ,  $K' > 0$  such that  $K \leq \log \|F'|\omega\| \leq K' \log |\omega|$  holds for every  $\omega \in \mathcal{Q}$ . Choose  $0 < \gamma < 1$  such that  $|\log |\omega|| \leq |\omega|^{-\gamma}$  holds for every  $\omega \in \mathcal{Q}$ . Then

$$K \leq \int \log |F'| d\nu \leq K \sum_{\omega \in \mathcal{Q}} \int |\log |\omega|| (d\nu|\omega) \leq K \sum_{\omega \in \mathcal{Q}} |\omega|^{1-\gamma} = K \sum_{n>0} \sum_{\omega: R(\omega)=n} |\omega|^{1-\gamma},$$

which is finite by (c) in Proposition 4.1. As

$$(13) \quad \mu = \frac{1}{\int R d\nu} \sum_{\omega \in \mathcal{Q}} \sum_{i=0}^{R(\omega)-1} (f_*^i) \nu|_{\omega},$$

we have

$$\begin{aligned} 0 < \int \log |F'| d\nu &= \int \sum_{i=0}^{R(x)-1} \log |f'(f^i x)| d\nu(x) = \sum_{\omega \in \mathcal{Q}} \sum_{i=0}^{R(\omega)-1} \int_{\omega} \log |f'(f^i x)| d\nu(x) \\ &= \sum_{\omega \in \mathcal{Q}} \sum_{i=0}^{R(\omega)-1} \int \log |f'| d((f_*)^i \nu|_{\omega}) = \int R d\nu \int \log |f'| d\mu < \infty. \end{aligned}$$

The desired result follows.  $\square$

## 5. ENTROPY FORMULA

In this last section we prove an entropy formula, connecting the metric entropy to the Lyapunov exponent. Although this formula is known to hold for a broad class of maps with critical and singular points, circle maps with logarithmic singularities have not been treated.

**Theorem 5.1.**  $h(f, \mu) = \int \log |f'| d\mu$ , where  $h(f, \mu)$  denotes the metric entropy.

A proof of this theorem uses Mănă's argument [15] that is outlined as follows. Define a family  $\{\rho_\beta\}_{\beta \in (0, \delta)}$  of functions on  $S^1$  by:  $\rho_\beta(x) = d_C(x)$  if  $d_C(x) \leq \beta$ ,  $\rho_\beta(x) = d_S(x)$  if  $d_S(x) \leq \beta$ , and  $\rho_\beta(x) = \beta$  in all other cases. Obviously  $0 < \rho_\beta \leq \beta$  holds, and Lemma 2.1 and Lemma 4.1 give  $\int -\log \rho_\beta d\mu < \infty$ . Let  $B(x, \rho_\beta; n) := \{y \in S^1 : |f^i x - f^i y| < \rho_\beta(f^i x) \ 0 \leq i < n\}$ . From [15], for  $\mu$ -a.e.  $x \in S^1$ ,

$$\sup_{\beta>0} \limsup_{n \rightarrow \infty} -\frac{1}{n} \log \mu(B(x, \rho_\beta; n)) = h(f, \mu).$$

Fix  $\kappa > 0$  so that  $Z = \{y \in S^1 : \frac{d\mu}{d\text{Leb}}(y) \geq \kappa\}$  has positive Lebesgue measure. We show that, for any  $\beta$  and a.e.  $x \in Z$ , the  $\limsup$  converges to  $\bar{\lambda} = \int \log |f'| d\mu$ .

**5.1. A lemma.** For  $x \in S^1$  and  $n > 0$ , let  $J_n(x) = [x - D_n(x), x + D_n(x)]$ . Let  $\beta \cdot J_n(x)$  denote the interval of length  $2\beta D_n(x)$  centered at  $x$ .

**Lemma 5.1.** For any  $x \in S^1$  and  $n > 0$  we have:

- (a)  $B(x, \rho; n) \supset \beta \cdot J_n(x)$ ;
- (b) if  $n$  is a deep return time of  $x$ , then  $B(x, \rho; n+1) \subset J_n(x)$ .



*Proof.* Lemma 2.2 gives  $|f^i(x) - f^i(y)| \leq \frac{2\beta}{\sqrt{L}} d_C(f^i(x)) d_S(f^i(x)) < \beta$  for all  $y \in \beta \cdot J_n(x)$  and every  $0 \leq i < n$ . If  $\rho_\beta(f^i(x)) < \beta$ , then  $|f^i(x) - f^i(y)| < \beta(f^i(x))$ . This proves (a). (b) follows from Lemma 2.5 and the bounded distortion of  $f^n|_{J_n(x)}$  from Lemma 2.2.  $\square$

**5.2. Upper estimate.** From the ergodic theorem, there exists a sequence  $(\theta_k)_k$  of positive numbers such that  $\theta_k \rightarrow 0$  and the following holds for each  $\theta_k$ : for any small  $\epsilon > 0$  and  $\mu$ -a.e.  $x$ , there exists  $n(x)$  such that for all  $n \geq n(x)$ ,  $\min\{d_C(f^n x), d_S(f^n x)\} \geq e^{-(\theta_k + \epsilon)n}$ . Also, for any small  $\epsilon > 0$  and  $\mu$ -a.e.  $x$  there exists  $n'(x)$  such that for all  $n \geq n'(x)$ ,  $e^{(\bar{\lambda} - \epsilon)n} \leq |(f^n)'x| \leq e^{(\bar{\lambda} + \epsilon)n}$ . In view of the definition (2), for  $\mu$ -a.e.  $x$  and all large  $n$  depending on  $x$ ,

$$(14) \quad |(J_n(x))| \geq \frac{1}{\sqrt{L}} e^{-(\bar{\lambda} + 3\epsilon)n - 2\theta_k n}.$$

Choose  $x$  to be a Lebesgue density point of  $Z$ . As  $|J_n(x)| \rightarrow 0$ ,  $\mu(J_n(x)) \geq \kappa|J_n(x)|$  holds for all large  $n$ . (a) in Lemma 5.1 and (14) give

$$-\frac{1}{n} \log \mu(B(x, \rho; n)) \leq -\frac{1}{n} \log \mu(I_n(x)) \leq \bar{\lambda} + 4\epsilon + 2\theta_k.$$

Hence  $\limsup_{n \rightarrow \infty} -\frac{1}{n} \log \mu(B(x, \rho; n)) \leq \bar{\lambda} + 4\epsilon + 2\theta_k$  holds for  $\mu$ -a.e.  $x \in Z$ . As  $\theta_k$  and  $\epsilon$  can be made arbitrarily small, the desired upper estimate holds.

**5.3. Lower estimate.** Write  $J$  for  $J_n(x)$ . For  $y \in S^1$ , let  $A(y) = \#\{0 \leq i < R(y) : f^i(y) \in J\}$ . Let  $N > 0$  be a large integer. Let  $B(y) = \#\{0 \leq i < N : F^i(y) \in J\}$ . Let  $T(y) = \sum_{i=0}^{N-1} A(F^i y)$  and  $U(y) = \sum_{i=0}^{N-1} R(F^i y)$ . Clearly,  $T(y) \leq U(y)$  and  $T(y) \leq \max_{0 \leq i < N} \{R(F^i y)\} B(y)$  hold.

Let  $m = \frac{30\lambda n}{\lambda \ln L}$ . Let  $Y = \{y \in S^1 : U(y) \leq m\}$ . By the  $F$ -invariance of  $\nu$ ,

$$(15) \quad N \int A d\nu = \int T d\nu = \int_Y T d\nu + \int_{Y^c} T d\nu.$$

We estimate the first integral of the right-hand-side. Let  $X = \{y : |B(y)/N - \nu(J)| < \nu(J)\}$ . We have  $\int_{X \cap Y} S d\nu \leq \int_X m B d\nu$ , because  $T \leq mB$  holds on  $Y$ . The definition of  $X$  gives  $\int_X m B d\nu \leq K m N \nu(J)$ , where  $K$  is a uniform constant bounding the density of  $\nu$ . Hence

$$(16) \quad \int_Y T d\nu = \int_{X \cap Y} T d\nu + \int_{X^c \cap Y} T d\nu \leq K m N \nu(J) + m \nu(X^c).$$

The second term of the right-hand-side can be made arbitrarily small by making  $N$  large. Indeed, by the ergodic theorem for  $(F, \nu)$ ,  $B/N \rightarrow \nu(J)$  a.e. as  $N \rightarrow \infty$ . The convergence in probability gives  $\nu(X^c) \rightarrow 0$  as  $N \rightarrow \infty$ .

We now estimate the second integral of the right-hand-side of (15). For a given  $N$ -string  $(a_1, \dots, a_N)$  of positive integers, let  $R_{a_1 \dots a_N} = \{y \in S^1 : R(F^i y) = a_i, 1 \leq i \leq N\}$ . For each component  $Q$  of  $R_{a_1 \dots a_N}$ , (c) in Proposition 4.1 and the bounded distortion of  $F^{a_1 + \dots + a_{N-1}}|_Q$  give  $|\{y \in Q : R(F^{a_1 + \dots + a_{N-1}}(y)) = a_N\}| \leq 2\delta L^{-\frac{\lambda a_N}{26}} |Q|$ . Summing this over all components gives  $|R_{a_1 \dots a_N}| \leq L^{-\frac{\lambda a_N}{26}} |R_{a_1 \dots a_{N-1}}|$ , and therefore  $|R_{a_1 \dots a_N}| \leq L^{-\frac{\lambda}{26}(a_1 + \dots + a_N)}$ . This yields

$$\int_{Y^c} T d\nu \leq \int_{Y^c} U d\nu = \sum_{\substack{(a_1, \dots, a_N) \\ a_1 + \dots + a_N > m}} (a_1 + \dots + a_N) \nu(R_{a_1 \dots a_N}) \leq K \sum_{r > m} \sum_{\substack{(a_1, \dots, a_N) \\ a_1 + \dots + a_N = r}} L^{-\frac{\lambda r}{27}}.$$

As  $R > M_0$ ,  $\frac{N}{r} \leq \frac{1}{M_0}$  holds. By Stirling's formula for factorials, the number of all feasible  $(a_1, \dots, a_N)$  with  $a_1 + \dots + a_N = r$  is  $\leq e^{cr}$ , where  $c \rightarrow 0$  as  $L \rightarrow \infty$ . Hence

$$(17) \quad \int_{Y^c} T d\nu \leq KL^{-\frac{\lambda m}{28}} \leq K|J|.$$

The last inequality follows from (14) and the definition of  $m$ .

Plugging (16) (17) into the right-hand-side of (15), dividing the result by  $N$  and passing  $N \rightarrow \infty$  we obtain

$$(18) \quad \mu(J_n(x)) \int R d\nu = \int A d\nu \leq Km\nu(J_n(x)) = \frac{30K\bar{\lambda}}{\lambda \ln L} n|J_n(x)|.$$

The equality is from (13).

Choose  $(n_k)_k$  denote an increasing infinite sequence of deep return times of  $x$ . By the Poincaré recurrence,  $\mu$ -a.e. point in  $Z$  has such a sequence. Let  $\epsilon > 0$  be an arbitrary small number. For any sufficiently large  $n_k$ ,  $|J_{n_k}(x)| \leq e^{-(\bar{\lambda}-\epsilon)n_k}$  holds. For such  $n_k$ , (18) and (b) in Lemma 5.1 give

$$-\frac{1}{n_k+1} \log \mu(B(x, \rho; n_k+1)) \geq -\frac{1}{n_k+1} \log \mu(J_{n_k}(x)) \geq -\frac{\log n_k}{2(n_k+1)} + \frac{n_k}{n_k+1}(\bar{\lambda} - \epsilon).$$

Taking the limit  $k \rightarrow \infty$  gives  $\limsup_{n \rightarrow \infty} -\frac{1}{n} \log \mu(B(x, \rho; n)) \geq \bar{\lambda} - \epsilon$ . As  $\epsilon$  is arbitrary, the desired lower estimate holds. This completes the proof of Theorem 5.1.

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